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Mechanical Modelling for the Behaviour of the Metallic Plate under the Effect of Bending Loads

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Abstract

In this paper we established the basic equations of dependence between the stresses and displacements that satisfy the boundary conditions by mathematical modelling using global analysis through potential energy for elastic plates in flexible form under the effect of asymmetric loading applied to its surface using the Reissner theory. This work leads us at the end to obtain a general theory for studying accurately the elastic plates in structures, so we prevent these structures from failure that leads to accidents.

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1. INTRODUCTION

In this research, we introduce a comprehensive general study of the phenomena of the flexibility of elastic metallic plates in mathematical ways with accurate results in tackling these phenomena. The properties of flexibility in metallic plates is largely based on the plate thickness as subordinate to its other dimensions [1,2,3,4,5]. In flexible plates with perfect elasticity we distinguish the following:

1. The plates are thin with small displacements, in which case proximity theories of linear elasticity are used for studying them [6,7,8,9,10].

2. The plates are thin with large displacements, in which case non-linear elasticity equations are used for studying them [11,12,13,14].

3. The plates are thick, in which case proximity theories of linear elasticity are used for studying them [15,16,17,18,19]. The earliest study of the physical phenomenon of flexibility used the classical theory. This study

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led to the solution of problems with certain boundary conditions to solve Laplace equation with displacements W without charges on the plate.

$$D\Delta\left(\Delta W\right) = 0$$

Where Δ partial differential Laplace operator, and D rigidity factor of the plate material. This study, however, was not satisfactory as a lot of accidents of fracture appeared in mechanical structures. Such accidents made researchers seek better solutions with more accurate results. Reissner and his team suggested developing the classical theory. His study was confined to adding a new equation with function stresses K to Laplace equation with displacements W. This new equation is $a^2\,\Delta k + k = 0$, and he solved both equations.

$$D\Delta(\Delta W) = 0 \quad , \quad a^2\,\Delta k + k = 0 \qquad (1)$$

Where $a^2 = (2/5)h^2$, his the plate thickness.

Although Reissner's theory has made some success in studying flexibility of plates, it remained restricted to special cases in tackling some problems in his experimental study, especially in studying the flexibility of cracked plate of different fracture modes.

The main objective of this research is to find all the basic equations sufficient for studying metallic plates in flexibility position, and to find stress fields, moment fields, and shears tresses fields in the form of displacement function Wand stress function K, and to generalize the boundary conditions on the plate edges in mathematical modelling using boundary integration and field theory depending on the potential and deformation energy, taking into consideration admissible static stress fields. Our work in this research includes four stages.

1. In the first stage we introduce the linear elasticity equations. We state the potential energy theory which we are going to use to get the plate theory equations.
2. In the second stage, we suppose admissible static stress fields which fit the plate's geometrical form, and we, in mathematical ways, introduce moment equations and shear stresses.
3. In the third stage we calculate deformation energy and stress work connect to an admissible static field to get, by using boundary integrations, potential energy equations, and boundary conditions on the edges of the plate, taking into consideration the classical theory hypothesis.
4. In the fourth stage we determine accurately moment equations and shear stresses equations in the cases of charged and uncharged plates, taking into consideration Reissner's theory hypothesis. Then we determine tangent and normal stresses on the edges of the plate, along with boundary condition equations.

II. LINEARELASTICITY EQUATIONS

II.1.Deformation Field: Solids are deformed under the influence of exterior charges. Suppose $\vec{e}_1$ position vector at M of the solid before deformation and $\vec{e}_2$ position vector at M after deformation. Then the displacement vector $\vec{u}$ is given by $\vec{u}$ is $\vec{u}=\vec{e}_2-\vec{e}_1$.During deformation period, the distances between the solid points change. If each point of the solid is determined by Cartesian coordinates (x_1,x_2,x_3),this change is defined by deformation field:

$$\varepsilon_{ik}=(1/2)\{u_{i,k}+u_{k,i}+u_{j,i}\times u_{j,k}\}$$

By neglecting the non-linear terms in the previous expression: $u_{j,i}\cdot u_{j,k}$,we find that the deformation field takes the following form:

$$\varepsilon_{ik}=(1/2)\{u_{i,k}+u_{k,i}\}$$

By using gradation theory in the fields, the previous expression can be put in the form of deformation field with reference to displacement vector $\vec{u}$ [7, 8, 9, 10, 11, 12]:

$$\overline{\overline{\varepsilon}}\left(\vec{u}\right) = \left(1/2\right)\overline{\overline{\text{grad}}}\,(\vec{u}) + T\,\overrightarrow{\text{grad}}\,(\vec{u}) \;\; ; \; \vec{u} = \vec{e}_2 - \vec{e}_1 \qquad (2)$$

II.2. Stress Field: When the plate is deformed, the position of the solid is deformed, the positions of molecules are changed, and molecular forces extend to include equilibrium position. In elasticity, the molecular forces have an effect diameter between molecules from the distance between neighbouring molecules [16, 17, 20,21]. At each point M from the edge (Σ') of (S') (figure.1), these forces are of the form:

$$\overrightarrow{T_n}(M)=\bar{\bar{\sigma}}(M)\times\vec{n}(M) \qquad (3)$$

where $\vec{n}$ normal vector at M on (S').

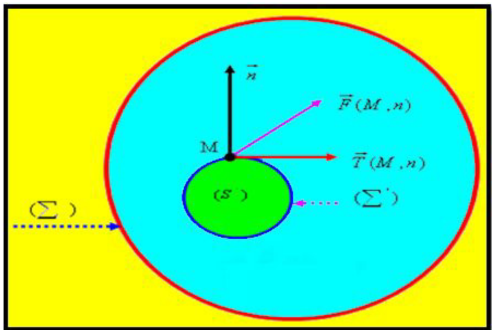

Fig. (1): a random part of the solid (s)

$\bar{\bar{\sigma}}(M)$ is called the stress field. By using divergence theory, the equilibrium equations are:

$$\bar{\bar{\sigma}}\cdot\vec{n}=\vec{F} \quad \text{in } (S) \quad , \quad \overrightarrow{\mathrm{div}}(\bar{\bar{\sigma}})+f\,\vec{f}\,' \quad \text{on } (\Sigma) \qquad (4)$$

where $\vec{f}\,'$ the mass forces and $\vec{F}$ the charges on the solid. In elasticity we depend on Hook's law:

$$\bar{\bar{\sigma}}(\vec{u})=\lambda(\mathrm{div}\,\vec{u})\bar{\bar{I}}+2\mu\bar{\bar{\varepsilon}}(\vec{u}) \ , \ \bar{\bar{\varepsilon}}=-(\nu/E)(\mathrm{tr}\,\bar{\bar{\sigma}})\bar{\bar{I}}+((1+\nu)/E)\bar{\bar{\sigma}} \qquad (5)$$

where E Young factor, ν Poisson's factor, λ and μ Lamy's constants.

II.3.Theory of Potential and Deformation Energy: Now we give the definition of deformation energy:

$$\eta(\bar{\bar{\sigma}})=\iiint_{(s)}\omega(M)\,dv \qquad (6)$$

Where $\omega(M)$ is given either by deformation only, by deformation and stresses, or by stresses only as in the following expression:

$$\left.\omega(M)=(1/2)\{\lambda(\varepsilon_{kk})^2+2\mu\varepsilon_{ij}\varepsilon_{ij}\} \ , \ \omega(M)=(1/2)\{((1+\nu)/E)\sigma_{ij}\sigma_{ij}-(\nu/E)(\sigma_{kk})^2 \ , \ \omega(M)=(1/2)\sigma_{ij}\varepsilon_{ij}\}\right\} (7)$$

By definition, the following expression

$$V(\bar{\bar{\sigma}}')=-W(\bar{\bar{\sigma}}')+\iint_{(\Sigma)}T_i'.\bar{U}_i\,d\sigma \qquad (8)$$

is called potential energy of an admissible static field $(\bar{\bar{\sigma}}')$ for a given problem. By definition $(\bar{\bar{\sigma}}')$ is an admissible stress field which achieves the equilibrium equation in region (S) and the conditions:

$$\sigma'_{ij}n_j=\bar{F}_i \quad , \quad T'_i=\bar{F}_i \qquad (9)$$

at each point of (Σ), where the charges $\bar{F}_i$ are given, $\vec{n}$ normal vector on (Σ) and oriented to the exterior. For defining double integration at (8), we suppose it is given at each point P of the (Σ), one or the other of the components $\bar{F}_i$ or $\bar{U}_i$.

III. STRESSESIN BENDINGPLATES.

III.1. Forces and Torque: We consider that on P_-P_+ of (S) of the plate (S) there is a stress field defined as (Fig. 2):

$$(\sigma\ ,\bar{\bar{\sigma}}\times\vec{n}) \ ; \ \vec{n}=(n_1\,,n_2\,,0) \qquad (10)$$

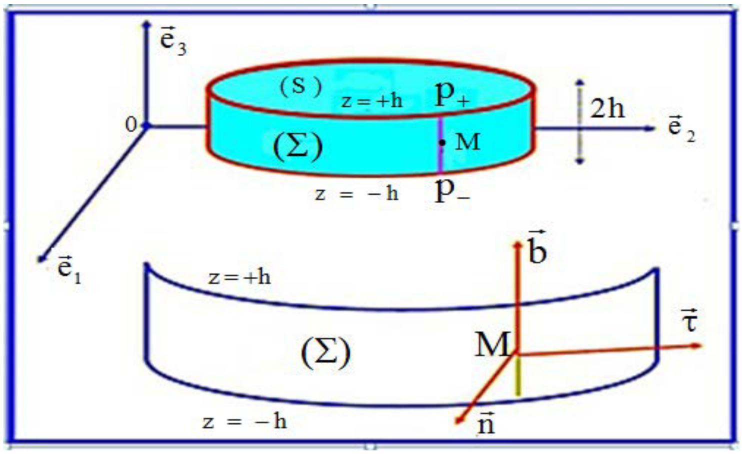

Fig. (2): An element of the elastic plate with the edge (Σ) at the coordinates (0,e$_1$,e$_2$,e$_3$)

The forces $\vec{R}$ are on the bending part of the plate

$$\vec{R} = \vec{r} + r\vec{z}\ ,\ \vec{R} = \int_{-h}^{+h} \overline{\overline{\sigma}} \cdot \vec{n}\, dz \quad ,\ R_i = (\int_{-h}^{+h} \sigma_{ij}\, dz) n_j ; i = 1,2,3 \qquad (11)$$

From expressions (11) we can calculate R_3, R_2, R_1 we find:

$$r = \vec{Q} \cdot \vec{n} \quad , \quad \vec{r} = \overline{\overline{N}} \cdot \vec{n} \qquad (12)$$

$\overline{\overline{N}}$ and $\vec{Q}$ are given by the expressions

$$N_{ij} = \int_{-h}^{h} \sigma_{ij}\, dz \quad , \quad Q_i = \int_{-h}^{h} \sigma_{i3}\, dz\ ;\ i = 1, 2 \qquad (13)$$

Q_1, Q_2 are shear stresses. The moments at point M (figure 2) are:

$$\overrightarrow{M} = \int_{-h}^{+h} \overrightarrow{MP} \times \left(\overline{\overline{\sigma}} \cdot \vec{n}\right) dz = \int_{-h}^{+h} z \cdot \vec{z} \times \left(\overline{\overline{\sigma}} \cdot \vec{n}\right) dz = \vec{z} \times \overline{\overline{M}} \cdot \vec{n}\ ,\ \overline{\overline{M}} = \int_{-h}^{h} z \cdot \overline{\overline{\sigma}}\, dz \Rightarrow M_{ij} = \int_{-h}^{h} z \cdot \sigma_{ij} dz\ ; i, j = 1,2 \qquad (14)$$

$\overline{\overline{M}}$ is the moment field at point M.

III.2 Determination of Stresses: We suppose that the plate is charged on Z=+h, and the stresses vary in a linear way on the plate thickness [4, 5, 12, 17, 19, 20, 21]. In this case we verify that $\overline{\overline{N}}$ and $\vec{r}$ are null. By doing integrations and calculations at (14) we get moments by stresses or stresses by moments as:

$$zM_{ij} = (2h^3/3)\sigma_{ij}\ ,\ \sigma_{ij} = (3/2h) M_{ij} z\ ;\ i,j=1,2 \qquad (15)$$

The free equilibrium equation at (S) by stress $\sigma_{ij,j}=0$ where i, j=1,2,3 gives three equilibrium equations in the following form:

$$(\sigma_{11})_x + (\sigma_{12})_y + (\sigma_{13})_z = 0\ ,\ (\sigma_{12})_x + (\sigma_{22})_y + (\sigma_{23})_z = 0\ ,\ (\sigma_{13})_x + (\sigma_{23})_y + (\sigma_{33})_z = 0 \qquad (16)$$

In the case of the charges P on Z=±h we distinguish two cases:

1") On Z=±h the charging conditions are:

$$\overline{\overline{\sigma}} \times \vec{z} = p\vec{z}\ ;\ (\sigma_{13} = \sigma_{23} = 0\ , \sigma_{33} = p) \qquad (17)$$

2") On $Z = -h$ the charging conditions are:

$$\overline{\overline{\sigma}} \cdot \vec{z} = \vec{0}\ ;\ (\sigma_{13} = \sigma_{23} = \sigma_{33} = 0) \qquad (18)$$

where P is the charge on the exterior surface Z=+h. from 1") and 2") it is obvious that on the surface Z=-h there are not stress fields σ_{13}, σ_{23}, σ_{33}. These results are in accordance with the experiments as the comparison shows. By doing calculations and integrations at (14), taking into consideration the conditions (16), we get the stresses σ_{13} and σ_{23} by the moments and the plate thickness:

$$\left.\sigma_{13}=\left[(M_{11})_x+(M_{12})_y\right]\left((3/4h)(1-z^2/h^2)\right),\ \sigma_{23}=\left[(M_{12})_x+(M_{22})_y\right]\left((3/4h)(1-z^2/h^2)\right)\right\}\quad (19)$$

By the boundary integrations calculations at (13), we get shear stresses Q_1 and Q_2 by the moments as follows:

$$\left.Q_1=(\partial M_{11}/\partial x)+(\partial M_{12}/\partial y)\ ,\ Q_2=(\partial M_{12}/\partial x)+(\partial M_{22}/\partial y)\right\}\quad (20)$$

Depending on (20) and (19), we can determine the stresses σ_{i3} by the shear stress Q_i, where we find that:

$$\sigma_{i3}=(3/4h)(1-z^2/h^2)Q_i\ ;\ i=1,2\quad (21)$$

Let us now calculate σ_{33}. From the expression (16) we calculate $\partial\sigma_{33}/\partial z$ we find:

$$(\partial\sigma_{33}/\partial z)=-(\partial\sigma_{13}/\partial x)-(\partial\sigma_{23}/\partial y)\quad (22)$$

By compensating σ_{13} and σ_{33} from (21) in (22) and doing the integrations, we get, after mathematical operations in integrations, σ_{13} as follows:

$$\sigma_{33}=-(3h/4)\left[(Q_1)_x+(Q_2)_y\right]\left(z+c-z^3/3h^2\right)\quad (23)$$

Where c is integration constant. Expression (23) can be put by the charges P. by using the charging conditions (17) and (18) we find, after mathematical operations, the basic equation by the charges P and the shear stresses Q_1 and Q_2.

$$(Q_1)_x+(Q_2)_y+P=0\quad (24)$$

For calculating the integration constant c, by considering the conditions on $Z=\pm h$ we find that $c=2h/3$. By compensating c in (23) and by compensating (24) in (23) we get σ_{33} in the following form:

$$\sigma_{33}=(3P/4h)\left[z-\left(z^3/3h^2\right)+\left(2h/3\right)\right]\quad (25)$$

III-3.The Nature of Charges on the Plate Edge: Let us now examine the nature of charges $\vec{F}$ compatible with the stress field $\overline{\overline{\sigma}}$ on the edge of the plate (Σ). According to expression (4) and the continuous milieu mechanics concept, we have:

$$\overline{\overline{\sigma}}\cdot\vec{n}=\vec{F}\left(=\vec{f}+f\cdot\vec{z}\right)\quad (26)$$

where $\vec{f}$ determined by the moment field and f by $\vec{Q}$ as follows:

$$\left.\vec{f}=\left(3/2h^3\right)\left(\overline{\overline{M}}\cdot\vec{n}\right)\cdot z\ ,\quad f=(3/4h)\left(\vec{Q}\cdot\vec{n}\right)\left(1-z^2/h^2\right)\right\}\quad (27)$$

Previously, we found that force vector $\vec{R}$ on p_-p_+ of the edge (Σ) is (Fig.2) : $\vec{R}=\left(\vec{Q}\cdot\vec{n}\right)\cdot\vec{z}$ and moment vector $\vec{M}=\vec{z}\times\overline{\overline{M}}\cdot\vec{n}$. The analysis of the problem which we did shows that the reduced elements at point M of the part p_-p_+ of (Σ) are $\vec{Z}\times\overrightarrow{M^*}$, $r^*\cdot\vec{Z}$ on the part of (Σ) where the charges $\overrightarrow{M^*}$ and r^* are given, we will have the following conditions:

$$\vec{Q}\cdot\vec{n}=r^*\quad ,\quad \overline{\overline{M}}\cdot\vec{n}=\overrightarrow{M^*}\quad (28)$$

IV. POTENTIAL ENERGY

IV.1. Calculating of Deformation Energy by an Admissible Static Field: In the deformation energy, we take the function $\omega(M)$ from expression (7) in the form of stress field only. Consequently, the deformation energy in expression (6) becomes as follows:

$$\eta\left(\bar{\bar{\sigma}}\right) = \iiint_{(S)} \omega\left(\bar{\bar{\sigma}}\right) dv = \iint_{\Omega}\left[\int_{-h}^{h} \omega\left(\bar{\bar{\sigma}}\right) dz\right] dx\,dy \qquad (29)$$

Let us now calculate the deformation energy in expression (29) by the field $\left(\bar{\bar{\sigma}}\right)$. In fact, the double integration taken on the part Ω at plane $Z = o$, and name $\partial\Omega$ edge of Ω. If we compensate $\omega\left(\bar{\bar{\sigma}}\right)$ by the form of stress field from expression (7) in (29), and did the mathematical operations for each term in (29), we find that the deformation energy is as follows:

$$\eta\left(\bar{\bar{\sigma}}\right) = \alpha_1 \iint_{\Omega}\left[(B_1)^2+\alpha_2(B_2)+\alpha_3-\alpha_4(B_1)+\alpha_5(B_3)^2+\alpha_5(B_4)\right] dx\,dy \; (30)$$

where:

$$M_x = M_{11}, M_y = M_{22}, M_{xy} = M_{12} \;,\; M_x + M_y = B_1, M_{xy}^2 - M_x M_y = B_2$$

$$(\partial M_x / \partial x) + (\partial M_{xy} / \partial y) = Q_x = B_3 \;,\; (\partial M_x y / \partial x) + (\partial M_y / \partial y) = Q_y = B_4 \; (31)$$

$$\left.\alpha_1 = 1/2D(1-\nu^2) \;,\; \alpha_2 = 2(1+\nu) \;,\; D = 2h^3E/3(1-\nu^2) \;,\; \alpha_3 = 52p^2h^4/105 \;,\; \alpha_4 = 4\nu p h^4/5 \;,\; \alpha_5 = 4(1+\nu)h^2/5\right\} \quad (32)$$

IV.2.Calculating of Stress Work by an Admissible Static Field: Given that $\left(\sum '\right)$ is part of the edge (Σ) where the displacements $\bar{\bar{U}}'$ are given, the work in this case is:

$$\xi = \iint_{\Sigma'} \vec{F}\overline{\overline{U}}'\, ds\, dz = \int_{\partial\Omega'} \left[\int_{-h}^{h} \vec{f}\vec{\bar{u}}\, dz + \int_{-h}^{h} f\,\bar{u}\, dz\right] ds \qquad (33)$$

where $\vec{F}$ the charges, $\overline{\overline{U}}'$ general vector of edge displacements of the plate according to the normal and the tangent. $\vec{f}$ and f are determined in expression (27), and $\vec{F}$ is determined in (26), and

$$\bar{\bar{U}}' = \bar{\bar{u}} + \bar{u}\cdot\bar{z} \qquad (34)$$

By compensating (27) in (33) we get, after calculation the work:

$$\left.\chi = \int_{\partial\Omega'}\left[\int_{-h}^{+h} \beta_1\left(\overline{\overline{M}}\cdot\vec{n}\right) z\cdot\bar{\bar{u}}\, dz + \int_{-h}^{+h} \beta_2\left(\overline{Q}\cdot\vec{n}\right)\beta_3\bar{u}\, dz\right] ds = \int_{\partial\Omega'}\left[\beta_1\left(\overline{\overline{M}}\cdot\vec{n}\right)\int_{-h}^{+h} z\cdot\bar{\bar{u}}\, dz + \beta_2\left(\overline{Q}\cdot\vec{n}\right)\int_{-h}^{h}\beta_3\bar{u}\right] ds\right\} \quad (35)$$

If we suppose in (35) that:

$$\beta_1 = 3/2h^3 \;,\; \beta_2 = 3/4h \;,\; \beta_3 = 1 - z^2/h^2$$

$$\bar{\bar{u}}' = \beta_1 \int_{-h}^{h} z\cdot\bar{\bar{u}}\, dz \;\;,\;\; \bar{u}' = \beta_2 \int_{-h}^{h} \beta_3\cdot\bar{u}\, dz \qquad (36)$$

we find that the work in (35) takes the following form:

$$\chi = \iint_{\Sigma'} \vec{F}\cdot\overline{\overline{U}}'\, ds\, dz = \int_{\partial\Omega'}\left[(\overline{\overline{M}}.\vec{n})\bar{\bar{u}}' + (\vec{Q}\cdot\vec{n})\bar{u}'\right] ds \qquad (37)$$

We now suppose that:

$$\left.M_n = \vec{n}\cdot\left(\overline{\overline{M}}.\vec{n}\right) \;,\; M_s = \vec{\tau}\cdot\left(\overline{\overline{M}}.\vec{n}\right) \;,\; Q_n = \vec{Q}\cdot\vec{n} \;,\; \bar{u}'_s = \bar{\bar{u}}'\,\vec{\tau} \;,\; \bar{u}'_n = \overline{\overline{u}'}\cdot\vec{n}\right\} \quad (38)$$

The expression (36) is used for calculating $\bar{u}', \bar{\bar{u}}'$ and leads to taking on $(\bar{\Sigma}')$ the following approaches:

$$\bar{\bar{u}}(s,z) = z.\bar{\bar{u}}'(s) \qquad , \qquad \bar{u}(s,z) = \bar{u}'(s)$$

From the previous approach expressions we find that the displacement general vector takes the following form:

$$\bar{\bar{U}}' = z.\bar{\bar{u}}'(s) + \bar{u}'(s).\vec{z}$$

By using (38) in (37) we get the stress-work expression more comprehensively and more accurately in results as follows:

$$x = \int_{\partial\Omega'} \left[M_n \bar{u}'_n + M_s \bar{u}'_s + Q_n \bar{u}'\right] ds \quad (39)$$

where $\bar{u}'_n$, $\bar{u}'_s$ the edge displacement on the part $\partial\bar{\Omega}'$ in the direction of the tangent and the normal.

IV.3. The Potential Energy Expression: By compensating the work calculated in expression (39) and the deformation energy in expression (30) in the potential energy, we get the potential energy expression of an admissible static field that is the most comprehensive and accurate in metallic plates:

$$V\left(\bar{\bar{\sigma}}\right) = -\alpha_1 \iint_{\Omega} \left\{(B_1)^2 + \alpha_2(B_2) - \alpha_3(B_1) + \alpha_4 + \alpha_5[(B_3)^2 + (B_4)^2]\right\} dxdy + \int_{\bar{\Omega}'} (M_n \bar{u}'_n + M_s \bar{u}'_s + Q_n \bar{u}') ds \qquad (40)$$

And the basic plate equation is given by the following:

$$(Q_x)_x + (Q_y)_y + p = 0 \Rightarrow (M_x)_{xx} + 2(M_{xy})_{xy} + (M_y)_{yy} + p = 0 \quad (41)$$

IV.4.Developing of the Classical Theory of Bending:.

IV.4.1Calculation of Stress Equations: For the classical theory to be applicable with good results in studying the fact that the plate thickness 2h is small in relation to the plate dimensions, the displacements of the points on the plate's surface are also small in relation to the thickness 2h.

To get all the equations sufficient for studying plates in the position of bending from the potential energy expression, we neglect all the terms containing h^2, h^3,…etc. In this hypothesis the potential energy expression (40), in the classical theory, becomes:

$$V\left(\bar{\bar{\sigma}}\right) = -\alpha_1 \iint_{\Omega} \left\{(B_1)^2 + \alpha_2(B_2)\right\} dxdy + \int_{\partial\bar{\Omega}'} (M_n \bar{u}'_n + M_s \bar{u}'_s + Q_n \bar{u}') ds \qquad (42)$$

Now, we can get Euler's equations by using multiplications of Lagrange W(x,y), where we find:

$$V(\bar{\bar{\sigma}}) = \iint_{\Omega} W\left[(M_x)_{xx} + 2(M_{xy})_{xy} + (M_y)_{yy} + p\right] dx\,dy \qquad (43)$$

Euler's equations as a whole lead us to finding moments expressions by displacements W. If we compensate (42) in (43) and do boundary integration in the region Ω, taking into consideration the conditions of limits, we get:

$$\left.M_x = -D\left[(1-\nu)(W)_{xx} + \nu\Delta W\right],\ M_y = -D\left[(1-\nu)(W)_{yy} + \nu\Delta W\right],\ M_{xy} = -D(1-\nu)(W)_{xy}\right\} \ (44)$$

The system of equations (44) can be put in the form of moment's field:

$$\bar{\bar{M}} = -D(1-\nu)\bar{\bar{W}} + \nu D(\Delta W)\bar{\bar{I}} \quad (45)$$

By deriving equations (44) in relation to x and y and by compensation in (41), we get the basic equation in studying the bending of plates by displacement W with the presence of charges P on the plate.

$$\frac{\partial 4W}{\partial x^4} + 2\frac{\partial 4W}{\partial x^2 \partial y^2} + \frac{\partial 4W}{\partial y^4} = P/D \Rightarrow \Delta(\Delta W) = P/D \qquad (46)$$

where Δ partial differential Laplace's operator. Also by compensating (44), after deriving it partially in relation to x and y, in shear stress expressions (20) we find, after calculation that the shear stresses by W are given in the following expressions:

$$Q_x = -D(\Delta W)_x \quad , \quad Q_y = -D(\Delta W)_y \qquad (47)$$

IV.4.2Generalizing of the Boundary Conditions in the Classical theory: We develop and generalize the boundary conditions in the classical theory by using Green's theory and gradation in the fields. According to Green's expression we have:

$$D\int_{\Omega} u\cdot\Delta(\Delta W)\,dx\,dy = -\iint_{\Omega} \bar{\bar{u}}'':\overline{\overline{M}}(W)\,dxdy + \int_{\partial\Omega}\{M_n(W)\cdot\frac{du}{dn} - [Q_n(W) + \frac{d}{ds}M_s(W)]\cdot u\}ds \quad (48)$$

In (48), $\bar{\bar{u}}'' = \overline{\overline{\text{grad}}}(\overrightarrow{\text{grad}}\,u)$ and $\bar{\bar{M}}$ moment's field by displacements W determined in (45). M_s, M_n, Q_n the normal and tangent stresses on the edge of the plate and are given, after calculations, by the following expressions:

$$\left.M_n = -D\{\Delta W - (1-\nu)\vec{\tau}\cdot\frac{d}{ds}\overrightarrow{\text{grad}}\,W\},\ M_s = -D(1-\nu)\vec{n}\cdot\frac{d}{ds}\overrightarrow{\text{grad}}W,\ Q_n = -D\frac{d}{dn}(\Delta W)\right\} \quad (49)$$

The boundary conditions for the problem of plates with the basic equation (46) in the most comprehensive and accurate form are:

$$M_n = \bar{M}_n,\ (dW/dn) = \bar{u}'_n\ ,\ Q_n + (dM_s/ds) = \bar{Q}_n + (d\bar{M}_s/ds)\ ,\ W = \bar{u}' \quad (50)$$

in (50) $\bar{M}_s, \bar{M}_n, \bar{Q}_n$ are the charges given on $\partial\bar{\Omega}$.Now, we present the following cases:

1. If the edge of the plate is uncharged, this edge does not undergo torsion moments and bending or vertical shear forces. In this case we have the following boundary conditions:

$$W=0\ ,\ (dW/dn)=0 \quad (51)$$

2. If the edge of the plate is fixed, this edge does not undergo rotation or vertical displacements. In this case we have the following boundary conditions:

$$M_n = 0\ ,\ Q_n + (dM_s/ds) = 0 \quad (52)$$

3. If the edge of the plate is pushed, in this case the edge of the plate undergoes rotation but without vertical displacements. The boundary conditions in this case are:

$$M_n = 0\ ,\ W = 0 \quad (53)$$

V. Modelling of Reissner's Theory of Bending Plates:

V.1. Euler's Equations: By using the potential energy expression (40) and taking multiplications of Lagrange into consideration, we can, by boundary integrations, find all the equations of the improved and sufficient. Reissner's theory for studying bending plates with the required adequacy in results and talking any posed problem in this concern. For this objective, we suppose.

$$J(\bar{\bar{\sigma}}) = V(\bar{\bar{\sigma}}) - V_1(\bar{\bar{\sigma}}) = V(\bar{\bar{\sigma}}) - \iint_{\Omega} W\left[(M_x)_{xx} + 2(M_{xy})_{xy} + (M_y)_{yy} + p\right]dx\,dy \quad (54)$$

We will transform the double integration on Ω in expression (54). Therefore, we divide the region (Ω) into two parts $\partial\bar{\Omega}$ and $\partial\bar{\Omega}'$ (fig.3) as follows:

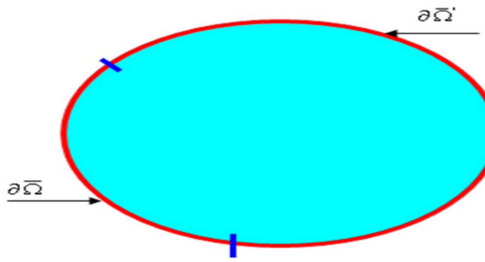

Fig. (3): $\partial\Omega = \partial\bar{\Omega}\cup\partial\bar{\Omega}'$

1. On the first part $\partial\bar{\Omega}'$, where the displacements: $\bar{u}'_n$، $\bar{u}'_s$ ،$\bar{u}'$ are given. In this case we determine the normal and tangent stresses $\bar{M}_n, \bar{M}_s, \bar{Q}_n$

2. On the second part $\partial\bar{\Omega}$ where the stresses $\bar{M}_n, \bar{M}_s, \bar{Q}_n$ are given, and we determine the displacements $\bar{u}'_n$، $\bar{u}'_s$ ،$\bar{u}'$ (Figure3) the geometrical position of the parts of the edge $\partial\Omega$. Therefore, we first transform the double integration in (54) to two integrations.

The first is on the part $\partial\Omega$, and the second is on the region (Ω). By this division we get the following expression.

$$\left.\iint_{\Omega} W\left(\operatorname{div}\overrightarrow{Q}+P\right) dx\, dy=\iint_{\Omega} W P\, dx\, dy+\int_{\partial\Omega} Q_n W\, ds-\iint_{\Omega}\overrightarrow{Q}\,\overrightarrow{\operatorname{grad}} W\, dx\, dy\right\} \quad (55)$$

To facilitate the mathematical operations in calculations, the third term in the second side in expression (55) can be written, according to Green's theory, by moments and displacements as follows:

$$\left.\iint_{\Omega}\overrightarrow{Q}\cdot\overrightarrow{\operatorname{grad}}(W)\, dx\, dy=\iint_{\partial\Omega}\left[\frac{dW}{dn}M_n+\frac{dW}{ds}M_s\right] ds-\iint_{\Omega}\left[M_x\frac{\partial 2W}{\partial x^2}+2\frac{\partial 2W}{\partial x\partial y}M_{xy}+M_y\frac{\partial 2W}{\partial y^2}\right] dx\, dy\right\} \quad (56)$$

If we take into consideration the division of edge $\partial\Omega$ and the comparison (56) with (55), we get the expression.

$$\iint_{\Omega} W\left(\operatorname{div}\overrightarrow{Q}+P\right) dx\, dy = \iint_{\Omega} W P\, dx\, dy + \int_{\partial\Omega} Q_n W\, ds-\int_{\partial\Omega}\left(M_n\frac{dW}{dn}+M_s\frac{dW}{ds}\right) ds+$$

$$+\iint_{\Omega}\left[M_x(W)_{xx}+2M_{xy}(W)_{xy}+M_y(W)_{yy}\right] dx\, dy \quad (57)$$

After this analysis, the expression (54) becomes in the most comprehensive and general form:

$$J\left(\overline{\overline{\sigma}}\right)=-\alpha_1\iint_{\Omega}\begin{bmatrix}(B_1)^2+\alpha_2(B_2)+\alpha_3-\alpha_4(B_1)\\ +\alpha_5(B_3)^2+\alpha_5(B_4)+\alpha_6+\alpha_7(B_5)\end{bmatrix} dx\, dy + \int_{\partial\overline{\Omega}'}\begin{bmatrix}\left(\overline{u}'_n+\frac{dW}{dn}\right)M_n+\left(\overline{u}'_s+\frac{dW}{ds}\right)M_s\\ -(W-\overline{u}')Q_n\end{bmatrix} d\, s$$

$$+\int_{\partial\overline{\Omega}}\left[\overline{M}_n\frac{dW}{dn}+\overline{M}_s\frac{dW}{ds}-\overline{Q}_n W\right] d\, s \quad (58)$$

$$B_5 = M_x (W)_{xx} + 2M_{xy}(W)_{xy} + M_y (W)_{yy} \;,\; \alpha_6 = 2D(1-\nu^2)pw \;,\; \alpha_7 = 2D(1-\nu^2)$$

Here we note that in expression (58) the performed transformations include the displacements $\overline{u}'_n$, $\overline{u}'_s$, $\overline{u}'$, because on the part $\partial\overline{\Omega}$,the values $\overline{M}_n , \overline{M}_s , \overline{Q}_n$ are given. Now we can use Euler's equations successfully to get the moments M_x , M_y , M_{xy} by the following:

$$\left.\begin{aligned}&M_x-\nu M_y-2(1+\nu)a^2(\partial Q_x/\partial x)-a^2\nu P=-D(1-\nu^2)(\partial 2W/\partial x^2)\\&M_y-\nu M_x-2(1+\nu)a^2(\partial Q_y/\partial y)-a^2\nu P=-D(1-\nu^2)(\partial 2W/\partial y^2)\;;\; a^2=2h^2/3\\&M_{xy}-a^2[(\partial Q_y/\partial x)+(\partial Q_x/\partial y)]=-D(1-\upsilon)(\partial 2W/\partial x\partial y)\end{aligned}\right\} \quad (59)$$

The expressions (59) determine the bending moments and torsion moments by Q_x , Q_y and w as follows:

$$\left.\begin{aligned}&M_x=2a^2(\partial Q_x/\partial x)-(\nu a^2P/(1-\nu)-D[(1-\nu)(\partial 2W/\partial x^2)+\nu\Delta W]\\&M_y=2a^2(\partial Q_y/\partial y)-(\nu a^2P/(1-\nu)-D[(1-\nu)(\partial 2W/\partial y^2)+\nu\Delta W]\\&M_{xy}=a^2[(\partial Q_y/\partial x)+(\partial Q_x/\partial y)]-D(1-\nu)(\partial 2W/\partial x\partial y);\; a^2=2h^2/3\end{aligned}\right\} (60)$$

From the previous expression, we conclude that Euler's equation by Lagrange multiplications W is the following equilibrium equation:

$$(M_x)_{xx} + 2(M_{xy})_{xy} + (M_y)_{yy} + p = 0 \qquad (61)$$

Besides this equation we add the following three equations of shear stresses by moments:

$$Q_x = (M_x)_x + (M_{xy})_y \quad (62-1)$$

$$Q_y = (M_{xy})_x + (M_y)_y \quad (62-2) \qquad (62)$$

$$(Q_x)_x + (Q_y)_y + P = 0 \quad (62-3)$$

The last equation (62-3) from (62) is merely the equilibrium equation (61), but in the form of shear stresses, where as (61) in the form of moments.

V.2. Determining of Stress Functions: Suppose p is the charges on the plate. Let us choose p_0 where:

$$p = \Delta p_0 = \mathrm{div}(\overline{\mathrm{grad}}\, p_0) \;,\; \overline{\mathrm{grad}}\, p_0 = \left(\partial p_0/\partial x \,,\, \partial p_0/\partial y\right) \quad (63)$$

Within this supposition, the basic equation (62-3) from (62) takes the following form:

$$(Q_x + \partial p_0 / \partial x)_x + (Q_y + \partial p_0 / \partial y)_y = 0 \quad (64)$$

Let G be a stress function where.

$$Q_x = (G)_y - (p_0)_x \;,\; Q_y = -(G)_x - (p_0)_y \quad (65)$$

By compensating (65) in (60) and doing the mathematical operations, we get the moments M_x, M_y, M_{xy} as functions of G, p_0, w as follows:

$$\left.\begin{aligned} &M_x = 2a^2(G)_{xy} - a^2(2-\nu)/(1-\nu)(p_0)_{xx} - (\nu a^2)(1-\nu)(p_0)_{yy} - D\left[(1-\nu)(\partial 2W/\partial x^2) + \nu\Delta W\right] \\ &M_y = -2a^2(G)_{xy} - a^2(2-\nu)/(1-\nu)(p_0)_{yy} - (\nu a^2)(1-\nu)(p_0)_{xx} - D\left[(1-\nu)(W)_{yy} + \nu\Delta W\right] \\ &M_{xy} = a^2\left[(G)_{yy} - (G)_{xx}\right] - 2a^2(p_0)_{xy} - D(1-\nu)(W)_{xy} \end{aligned}\right\} \quad (66)$$

We can write the shear stress equations Q_x, Q_y in (62) as functions of G and w with retaining the charges. If we compensate (66) in (62) we find, after mathematical operations we find the following equations:

$$\left. Q_x = a^2\Delta(G)_y - a^2(2-\nu)/(1-\nu)(p)_x - D(\Delta W)_x \;,\; Q_y = -a^2\Delta(G)_x - a^2(2-\nu)/(1-\nu)(p)_y - D(\Delta W)_y \right\} \quad (67)$$

By comparing (67) and (65) we get the following two expressions:

$$\left. a^2\Delta(G)_y - a^2(2-\nu)(1-\nu)(p)_x - D(\Delta W)_x = (G)_y - (p_0)_x - a^2\Delta(G)_x - a^2(2-\nu)(1-\nu)(p)_y - D(\Delta W)_y = -(G)_x - (p_0)_y \right\} \quad (68)$$

Expressions (68) can be written in a simpler form as following:

$$\left. -\partial\left(a^2\Delta G - G\right)/\partial y = \partial[p_0 - (2-\nu)a^2 p/(1-\nu) - D\Delta W]/\partial x \;,\; \partial\left(a^2\Delta G - G\right)/\partial x = \partial[p_0 - (2-\nu)a^2 p/(1-\nu) - D\Delta W]/\partial y \right\} (69)$$

In (69) if we suppose that:

$$\phi = p_0 - (2-\nu)(1-\nu)a^2 p - D\Delta W \;,\quad H = a^2\Delta G - G \qquad (70)$$

The expressions (69) become Cauchy-Remain's equations with the two analytical functions H and ϕ where $H + i\phi$ complex functions which achieve Cauchy-Remain's two conditions on the equations (69):

$$-(\partial H / \partial y) = (\partial \phi / \partial x) \;,\; (\partial H / \partial x) = (\partial \phi / \partial y) \quad (71)$$

By using the partial differential operator of Laplace Δ, and noticing that $\Delta\phi = o$ and $\Delta H = o$, the expression (70) takes the form following:

$$\left.\begin{aligned} &p - a^2\Delta p(2-\nu)/(1-\nu) - D\Delta(\Delta W) = 0 \quad (72\text{–}1) \\ &a^2\Delta(G+H) - (G+H) = 0 \qquad\qquad (72\text{–}2) \end{aligned}\right\} (72)$$

By writing $G + H = K$, the expression (72-2) becomes with the new function K as follows:

$$a^2\Delta k - k = 0 \quad (73)$$

where K is called the stress function. Equation (73), which we got mathematically by the previous complex analysis, is the Reissner equation suggested in his theory and added to Laplace's equation $D\Delta(\Delta w) = o$ in the classical theory to improve the performance of the classical theory in studying the plates bending, thus it is called Reissner's theory. Within the supposition $G + H = K$, (72) takes the following general form:

$$P-(2-\nu)(1-\nu)a^2\Delta P-D\Delta(\Delta W)=0\ ,\ \ \Delta k-(1/a^2)k=0 \qquad (74)$$

Based on this mathematical analysis we did, we can now put the moment equations M_x, M_y, M_{xy} and shear-stresses Q_x and Q_y in the most comprehensive and general form, and the most adequate in results, in studying metallic plates by stress function K and charges p.

By redoing the mathematical operations on (66) and (67) equations, we get the following general expressions in moments and shear stresses:

$$M_x=2a^2(K)_{xy}-2a^2[((2-\nu)/(1-\nu))a^2p+D\Delta W]_{xx}-[\nu a^2p/(1-\nu)]-D[(1-\nu)(\partial 2W/\partial x^2)+\nu\Delta W]$$

$$M_y=-2a^2(K)_{xy}-2a^2[((2-\nu)/(1-\nu))a^2p+D\Delta W)]_{yy}-[\nu a^2p/(1-\nu)]-D[(1-\nu)(\partial 2W/\partial y^2)+\nu\Delta W] \qquad (75)$$

$$M_{xy}=a^2[(K)_{yy}-(K)_{xx}-(2a^2)[((2-\nu)/(1-\nu))a^2p+D\Delta W)]_{xy}-[D(1-\nu)(W)_{xy}]$$

$$\left.Q_x=(K)_y-\left[a^2(2-\nu)(1-\nu)p+D\Delta W\right]_x\ ,\ Q_y=-(K)_x-\left[a^2(2-\nu)(1-\nu)p+D\Delta W\right]_y\right\}\ (76)$$

Here we remember that the moments and shears tresses $M_x, M_y, M_{xy}, Q_x, Q_y$ in (75) and (76) are applicable in the case of charges p but not p_o.

V.3. Making Sure of Accuracy of our expressions (75) and (76): In the special Problem, if the plate is uncharged on the surface $z=+h$, the moment expression (75) and the shear stresses (76) take the following form:

$$\left.\begin{aligned}
&M_x=2a^2(\partial 2K/\partial x\partial y)-2a^2\partial 2(D\Delta W)/\partial x^2-D\left[(1-\nu)(\partial 2W/\partial x^2)+\upsilon\Delta W\right]\\
&M_y=-2a^2[(\partial 2K/\partial x\partial y)+\partial 2(D\Delta W)/\partial y^2]-D\left[(1-\nu)(\partial 2W/\partial y^2)+\nu\Delta W\right]\\
&M_{xy}=a^2(\partial 2K/\partial y^2)-(\partial 2K/\partial x^2)-(2a^2)\partial 2(D\Delta W/\partial x\partial y)-D(1-\nu)(\partial 2W/\partial x\partial y)
\end{aligned}\right\}\ (77)$$

$$Q_x=(K)_y-(D\Delta W)_x\ ,\ Q_y=-(K)_x-(D\Delta W)_y \qquad (78)$$

Expressions (77) and (78) are the expressions of Reissner's in his experimental tackling of metallic plates in the special problems; we got Reissner's expressions from our general expressions (75) and (76). If we put in our expressions (75) and (76) $p=o$ and $K=o$, we can easily get the equations of the classical theory. The previous comparison between our results in (75) and (76) with the results of the classical theory and Reissner's theory proves the validity, accuracy and comprehensiveness of our theory in studying any posed problem in metallic plates and under any boundary conditions.

V.4. Generalizing of Boundary Conditions on the Plate's Edge and Determining of Normal and Tangent Stresses by Displacements W and K: Let us consider S as a bending part of $\partial\Omega$, and $\vec{t}$ tangent vector at M from S with the compounds $(\frac{dx}{ds}, \frac{dy}{ds}, 0)$ and $\vec{n}$ normal vector at this point with compound $(\frac{dy}{ds}, -\frac{dx}{ds}, 0)$.

From the definition of shear stresses, tangent and normal, we have:

$$Q_n=\vec{Q}\cdot\vec{n}=Q_x(dy/ds)-Q_y(dx/ds)\ ,\ Q_s=\vec{Q}\cdot\vec{\tau}=Q_x(dx/ds)+Q_y(dy/ds) \qquad (79)$$

By compensating Q_x, Q_y from (76) in (79) and doing mathematical operations we get the expressions Q_n and Q_s:

$$Q_n=\frac{dK}{ds}-\frac{d}{dn}[\frac{2-\nu}{1-\nu}a^2p+D\Delta W]\ ,\ Q_s=-\frac{dK}{dn}-\frac{d}{ds}[\frac{2-\nu}{1-\nu}a^2P+D\Delta W] \quad (80)$$

By using the same analysis we get the moments M_n and M_s:

$$M_n=\vec{n}\cdot(\overline{\overline{M}}\cdot\vec{n})=M_x(dy/ds)^2+M_y(dx/ds)^2-2M_{xy}(dx/ds)(dy/ds) \qquad (81)$$

$$M_s = \vec{t}\cdot(\overline{\overline{M}}\cdot\vec{n}) = \left(M_x - M_y\right)(dx/ds)(dy/ds) + M_{xy}\left[(dy/ds)^2-(dx/ds)^2\right] \qquad (82)$$

By using the moment expression from (75) in (81) and (82) and doing the mathematical operations by using gradation theory, we find that expressions (81) and (82) take the most comprehensive and accurate form in studying metallic plates:

$$M_n=-D\Delta W-\frac{2-\nu}{1-\nu}a^2p+2a^2\vec{\tau}\frac{d}{ds}\overrightarrow{grad}[\frac{2-\nu}{1-\nu}a^2pK+D\Delta W]+2a^2\vec{n}\frac{d}{ds}\overrightarrow{grad}\,K+D(1-\nu)\cdot\vec{\tau}\frac{d}{ds}\overrightarrow{grad}W]\Big\} \quad (83)$$

$$M_s=-K+2a^2\vec{\tau}\frac{d}{ds}\overrightarrow{grad}K-D(1-\nu)\vec{n}\frac{d}{ds}\overrightarrow{grad}\,W-2a^2\vec{n}\frac{d}{ds}\overrightarrow{grad}\,[\frac{2-\nu}{1-\nu}a^2p+D\Delta W]\Big\} \quad (84)$$

In our mathematical operations for determining M_n, M_s, Q_n, Q_s, we took into consideration the expressions (74). Expressions (80) and (82) are the most general in boundary conditions and include the study of any problem posed in talking charged plates in the case of bending.

V.5. Determining Boundary Conditions Sufficient for the Problem of Bending Plates: To get the boundary conditions sufficient for the problem of plates we reuse expression (58) and calculate only the transformation of $J(\bar{\bar{\sigma}})$ related to boundary integrations. We suppose F a function determined by the following expression:

$$F=-\ell\left\{(M_x+M_y)^2+2(1-\nu)(M_{xy}^2-M_xM_y)+(52/105)P^2h^4-(4/5)\nu h^2P(M_x+M_y)+2D(1-\nu^2)PW\right\}$$
$$+2D(1-\nu^2)\left\{M_x(W)_{xx}+2M_{xy}(W)_{xy}+M_y(W)_{yy}\right\}$$
$$+(4/5)(1+\nu)h^2\left\{((M_x)_x+(M_{xy})_y)^2+((M_{xy})_x+(M_y)_y)^2\right\}\;;\;\ell=1/2D(1-\nu^2) \quad (85)$$

Now we apply the form of the operator δ trans-formations to expression (58) and find:

$$\delta J\left(\bar{\bar{\sigma}}\right)=\iint_{\Omega}[F]\,\delta u\,dx\,dy+\int_{\delta\Omega}\left[F_{u_x}(dy/ds)+F_{u_y}(dx/ds)+[\phi]_u\right]\delta u\,ds \quad (86)$$

Where $[\phi]_u$ is given in the following expression:

$$[\phi]_u=\phi_u-d\phi_{u_s}/ds \quad , \quad u_s=du/ds \qquad (87)$$

and ϕ is the side term in expression (58). Now, taking into consideration Euler's equations, the transformations $J\left(\bar{\bar{\sigma}}\right)$ in (58) by the operator δ for the part $\partial\bar{\Omega}'$ are:

$$\delta J\left(\bar{\bar{\sigma}}\right)=\mu\int_{\partial\Omega'}\begin{bmatrix}Q_x\,\delta M_x(\frac{dy}{ds})-Q_y\delta M_y(\frac{dx}{ds})+\\ [Q_y\frac{dy}{ds}-Q_x\frac{dx}{ds}]\delta M_{xy}]\end{bmatrix}ds+\int_{\partial\Omega'}\begin{bmatrix}\left(\bar{u}_n'+\frac{dW}{dn}\right)\delta M_n+\left(\bar{u}_s'+\frac{dW}{ds}\right)\delta M_s-\\ -\left(W-\bar{u}'\right)\delta Q_n\end{bmatrix}ds \quad (88)$$

However, we know that:

$$Q_x=Q_n\frac{dy}{ds}+Q_s\frac{dx}{ds}\;,\;Q_y=Q_s\frac{dy}{ds}-Q_n\frac{dx}{ds} \qquad (89)$$

By compensating (89) in (88) and doing mathematical operations we find:

$$\delta J\left(\bar{\bar{\sigma}}\right)=\mu\int_{\partial\Omega'}\begin{bmatrix}Q_n[\delta M_x(\frac{dy}{ds})^2+\delta M_y(\frac{dx}{ds})^2-2\frac{dy}{ds}\frac{dx}{ds}\delta M_{xy}]\\ +Q_s[(\frac{dy}{ds}\frac{dx}{ds}\delta M_x-\frac{dy}{ds}\frac{dx}{ds}\delta M_y\\ +(\frac{dy}{ds})^2\delta M_{xy}+(\frac{dx}{ds})^2\delta M_{xy}]\end{bmatrix}ds+\int_{\partial\Omega'}\begin{bmatrix}\left(\bar{u}_n'+\frac{dW}{dn}\right)\delta M_n+\left(\bar{u}_s'+\frac{dW}{ds}\right)\delta M_s-\\ -\left(W-\bar{u}'\right)\delta Q_n\end{bmatrix}ds \qquad (90)$$

By applying the operator δ to (81) and (82) we find:

$$\left.\begin{aligned}\delta M_n&=(\delta\overline{\overline{M}}\cdot\vec{n})\cdot\vec{n}=\delta M_x\left(\frac{dy}{ds}\right)^2+\delta M_y\left(\frac{dx}{ds}\right)^2-2\delta M_{xy}\frac{dy}{ds}\cdot\frac{dx}{ds}\\ \delta M_s&=(\delta\overline{\overline{M}}\cdot\vec{\tau})\cdot\vec{n}=\delta M_x\frac{dx}{ds}\cdot\frac{dy}{ds}+\delta M_{xy}\left(\frac{dy}{ds}\right)^2-\delta M_{xy}\left(\frac{dx}{ds}\right)^2-\delta M_y\frac{dy}{ds}\cdot\frac{dx}{ds}\end{aligned}\right\}\quad(91)$$

By compensating (91) in expression (90) and doing the mathematical operations, we get the following expression in the form:

$$\delta J\left(\overline{\overline{\sigma}}\right)=\frac{-4h^2}{5D(1-\nu)}\int_{\partial\Omega'}\left\{\begin{Bmatrix}Q_n\delta M_n\\+Q_s\delta M_s\end{Bmatrix}ds\right\}ds+\int_{\partial\Omega'}\left\{(\bar{u}_n'+\frac{dW}{ds})\partial M_n+(\bar{u}_s'+\frac{dW}{ds})\delta M_s-(W-\bar{u}')\delta Q_n\right\}ds\quad(92)$$

Or in the form:

$$\delta J\left(\overline{\overline{\sigma}}\right)=\int_{\partial\Omega'}\left\{[(\bar{u}_n'+\frac{dW}{dn})-\frac{6(1+\upsilon)}{5Eh}Q_n]\delta M_n+[(\bar{u}_s'+\frac{dW}{ds})-\frac{6(1+\upsilon)}{5Eh}Q_s]\delta M_s-(W-\bar{u}')\,\delta Q_n\right\}ds\quad(93)$$

Expression (93) allows us to get the general, and comprehensive boundary conditions on the part $\partial\bar{\Omega}'$:

$$\frac{dW}{dn}-\frac{6}{5}\frac{1+\nu}{Eh}Q_n=-\bar{u}'_n\ ,\ \frac{dW}{ds}-\frac{6}{5}\frac{1+\nu}{Eh}Q_s=-\bar{u}'_s\ ,W=\bar{u}'\quad(94)$$

From expression (94) we notice that we have got three sufficient boundary conditions instead of two boundary conditions in the classical theory. In expression (94) $\bar{u}'$ is the medial displacements given, $\bar{u}_n'$ and $\bar{u}_s'$ are the medial rotational displacements of the edge of the plate according to the normal $\vec{n}$ and the tangent $\vec{t}$.

If the plate has a fixed edge, the edge displacements $\bar{u}_s',\bar{u}_n',\bar{u}'$ are null, and the expression (94) in this case gives the following three boundary conditions:

$$\frac{dW}{dn}-\frac{6}{5}\frac{1+\nu}{Eh}Q_n=0\quad,\quad\frac{dW}{ds}-\frac{6}{5}\frac{1+\nu}{Eh}Q_s=0\qquad W=0\qquad(95)$$

The boundary conditions by the stresses on the part $\partial\bar{\Omega}$ are:

$$M_n=\overline{M}_n\quad,\quad M_s=\overline{M}_s\quad,\quad Q_n=\overline{Q}_n\qquad(96)$$

Finally, by the work we performed in this research, we have got general comprehensive theory containing all expressions and boundary conditions which enable researchers to tackle any problem in bending metallic plates whether they be charged or uncharged and under whatever conditions on the plated. Our Coherent theory presents good results compared with the experimental studies in this area.

CONCLUSION

The work we did in this paper using the technique of mathematical modelling with high-level mathematical methods and the concept of continuous milieu mechanics has enabled us to get all the basic equations and expression, which link stress, moments and shears tresses, with their geometrical positions to the vertical displacements W and edge displacements as well as the boundary conditions, satisfactorily and comprehensively for studying bending metallic plates in the required accuracy. This work has been done by means of potential energy and deformation energy, as well as stress work in the metallic plate, using boundary integrations, field theory, and Lagrange theory by the operator δ to get Euler's equations by general moments and shear stresses, supposing admissible, static stress fields and the hypotheses of elasticity, the classical theory and Reissner's theory. This satisfactory work has led us to form a general theory with its expressions and boundary conditions which is sufficient to all what is needed for the study of metallic plates, charged and uncharged. The comparison between our results and the results of the classical and Reissner's theories shows that our results are more comprehensive and more general in the field of studying bending plates. The work in this research included four basic stages to give more accuracy and more comprehensiveness results. Finally, the work in this research has led us, in its results and by comparison with other results, to finding a coherent theory in all its expressions and boundary conditions to study any problem posed in the field of bending plates in structures mechanics and enable researchers to get the most accurate results. This coherent theory also allows, with the required accuracy studying cracked bending plates in the different modes of fracture.

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